

# A New Superlinearly Convergent Algorithm of Combining QP Subproblem with System of Linear Equations for Nonlinear Optimization<sup>☆</sup>

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## Abstract

In this paper, a class of optimization problems with nonlinear inequality constraints is discussed. Based on the ideas of sequential quadratic programming algorithm and the method of strongly sub-feasible directions, a new superlinearly convergent algorithm is proposed. The initial iteration point can be chosen arbitrarily for the algorithm. At each iteration, the new algorithm solves one quadratic programming subproblem which is always feasible, and one or two systems of linear equations with a common coefficient matrix. Moreover, the coefficient matrix is uniformly nonsingular. After finite iterations, the iteration points can always enter into the feasible set of the problem, and the search direction is obtained by solving one quadratic programming subproblem and only one system of linear equations. The new

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algorithm possesses global and superlinear convergence under some suitable assumptions without the strict complementarity. Finally, some preliminary numerical experiments are reported to show that the algorithm is promising.

*Keywords:* Nonlinear optimization, Sequential quadratic programming, Method of strongly sub-feasible directions, Global convergence, Superlinear convergence

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## 1. Introduction

In this paper, we consider the following nonlinear inequality constrained optimization problem

$$\begin{aligned} \text{(NCP)} \quad & \min f_0(x) \\ & \text{s.t. } f_j(x) \leq 0, \quad j \in I \triangleq \{1, 2, \dots, m\}, \end{aligned}$$

where  $x \in R^n$  and the functions  $f_j(x) : R^n \rightarrow R$  ( $j \in \{0\} \cup I$ ) are all continuously differentiable. We denote the feasible set and gradients for problem (NCP) as follows

$$X = \{x \in R^n : f_j(x) \leq 0, j \in I\}, \quad g_j(x) = \nabla f_j(x), \quad j \in \{0\} \cup I.$$

It is well-known that sequential quadratic programming (SQP) algorithms are acknowledged to one of successful algorithms available for solving problem (NCP) and have good superlinear convergence properties, they have been widely studied and investigated by many authors in [1, 2, 3, 4].

The iterative process of the standard SQP algorithms is as follows. Let the current iteration point be  $x$ . Computing a search direction  $\bar{d}$  by solving

the following quadratic programming (QP) subproblem

$$\begin{aligned} \min \quad & g_0(x)^T d + \frac{1}{2} d^T B d \\ \text{s.t.} \quad & f_j(x) + g_j(x)^T d \leq 0, \quad j \in I, \end{aligned}$$

where  $B \in R^{n \times n}$  is a symmetric matrix that approximates the Hessian of the Lagrangian function associated with problem (NCP) at  $(x, \lambda)$ , and  $\lambda$  is a vector of nonnegative Lagrange multiplier estimates. Perform a search to determine a steplength  $t$  and let the next iteration point be  $\bar{x} = x + t\bar{d}$ .

However, in the classical SQP algorithms, there are two shortcomings: (i) (QP) subproblem may be inconsistent, i.e., the feasible set of (QP) subproblem may be empty; (ii) the Maratos effect [5] may occur, i.e., a full step of one can not be taken close to a solution of problem (NCP). In order to overcome disadvantage (i), various techniques have been proposed in [1, 6]. A popular way to overcome disadvantage (ii) is to use a higher-order direction, which is generated by solving a (QP) subproblem [1] or a system of linear equations (SLE) [7], or directly given by an explicit formula [8].

For SQP algorithms, feasible SQP (FSQP) algorithms are particularly useful for solving those problems arising from engineering design where the objective function  $f_0$  might be undefined outside the feasible set  $X$ . Another advantage of FSQP algorithms is that the objective function  $f_0$  can be used as a merit function to avoid the use of a penalty function. In particular, Panier and Tits [1] present a FSQP algorithm in which the generated iteration points lie in the feasible set  $X$ . Two or three (QP) subproblems need to be solved at each iteration. In order to obtain the global convergence, they need to strengthen the requirement on the first-order feasible descent condition. The superlinear convergence rate is proved under the strict com-

plementarity assumption. Zhu and Jian [9] further improve the algorithm in [1]. They introduce a new definition for the first-order feasible condition which is weaker than the first-order feasible descent condition in [1], and propose a new FSQP algorithm based on this new condition. The strict complementarity assumption is also necessary for obtain the superlinear convergence.

One shortcoming of FSQP algorithms is that they are usually require a feasible starting point, while computing such a point is generally a nontrivial work [10]. In order to overcome this shortcoming, Polak et al. [11] propose a combined phase I-phase II algorithm with arbitrary initial point for solving problem (NCP). This algorithm becomes a method of feasible directions (MFD) [12] when iteration points enter into the feasible set  $X$ . Jian further improve algorithm [11] and propose a method of strongly sub-feasible direction in [13], which not only unified automatically the operations of minimization (Phase I), but also guaranteed that the number of the functions satisfying the inequality constraints is nondecreasing. Since their algorithms only using the information of first-order derivatives, the algorithms in [11, 13] converge linearly at best.

In this paper, motivated by the ideas in [9, 13], we propose a new algorithm combining (QP) subproblem with method of strongly sub-feasible directions for solving problem (NCP). Unlike algorithm in [9], a descent direction  $d_0$  for  $f_0$  at  $x$  is obtained by solving a (QP) subproblem which is always feasible. For obtaining the global convergence, a mere feasible direction  $\tilde{d}$  is obtained by solving a SLE. Then,  $d_0$  is tilted by making a convex combination  $q = (1 - \beta)d_0 + \beta\tilde{d}$  of  $d_0$  and  $\tilde{d}$ , where  $\beta$  satisfies certain con-

dition. With the help of this convex combination, the global convergence is proved. In order to overcome the Maratos effect and obtain the superlinear convergence, a higher-order correction direction  $d_1$  is computed by a new SLE which has a common coefficient matrix with the previous SLE. Under the strong second-order sufficient conditions without the strict complementarity, the new algorithm is proved to be superlinearly convergent. Moreover, the initial iteration point is chosen arbitrarily, and after finite iterations, the iteration points can always enter into the feasible set  $X$ . Finally, some numerical results are reported to shown that the proposed algorithm is promising.

At the end of this section, the main features of the proposed algorithm are summarized as follows:

- the initial iteration point is arbitrary, and the number of satisfied constraint functions is nondecreasing;
- the objective function of problem (NCP) is used directly as the merit function, and the line search techniques are different from others;
- at each iteration, the search direction is generated by solving one QP subproblem and one or two SLEs with the same coefficient matrix;
- after finite iterations, the search direction is obtained by solving one QP subproblem and only one SLE, and the iteration points always enter into the feasible set  $X$ ;
- under some mild conditions without the strict complementarity, the proposed algorithm possesses global and superlinear convergence.

This paper is organized as follows. In the next Section 2, we present the details of our algorithm and discuss its properties. In Sections 3 and 4, the

algorithm is proved to possess global and superlinear convergence, respectively. In Section 5, some elementary numerical experiments are reported. Finally, some concluding remarks are given in Section 6.

## 2. Description of algorithm

To simplify the analysis, we use the following notations

$$I^-(x) = \{j \in I : f_j(x) \leq 0\}, \quad I^+(x) = \{j \in I : f_j(x) > 0\}, \quad (1)$$

$$\varphi(x) = \max\{0, f_j(x), j \in I\} = \max\{0, f_j(x), j \in I^+(x)\}, \quad (2)$$

$$\bar{f}_j(x) = \begin{cases} f_j(x), & j \in I^-(x); \\ f_j(x) - \varphi(x), & j \in I^+(x), \end{cases} \quad I(x) = \{j \in I : \bar{f}_j(x) = 0\}. \quad (3)$$

Assume that the following two basic assumptions for problem (NCP) are hold throughout this paper.

**Assumption 2.1.** *The functions  $f_j$  ( $j \in \{0\} \cup I$ ) are all first-order continuously differentiable.*

**Assumption 2.2.** *The gradient vectors  $\{g_j(x) : j \in I(x)\}$  are linearly independent for each  $x \in R^n$ .*

**Remark 2.1.** *In Assumption 2.2, the linearly independent gradients contains two parts: the one part is the gradients of the feasible constraint functions in active set, and the other part is the gradients of the maximal violated functions. This assumption is becoming the standard linearly independent constraint qualification (LICQ) only if the iteration point is feasible. Moreover, this assumption plays a big role in the analysis of the following Lemmas 2.2, 2.3 and Theorem 3.1.*

For the current iteration point  $x^k$  and an associated symmetric positive definite matrix  $B_k$ , using the notations above, we introduce the following (QP) subproblem [14]

$$\begin{aligned} \text{(QPs)} \quad & \min \quad g_0(x^k)^T d + \frac{1}{2} d^T B_k d \\ & \text{s.t.} \quad \bar{f}_j(x^k) + g_j(x^k)^T d \leq 0, \quad \forall j \in I, \end{aligned}$$

and we denote simply

$$I_k^- = I^-(x^k), \quad I_k^+ = I^+(x^k), \quad I_k = I(x^k), \quad \varphi_k = \varphi(x^k).$$

It is obviously that subproblem (QPs) has the following merits:

- (i) subproblem (QPs) is always feasible with feasible solution  $d = 0$ ;
- (ii) subproblem (QPs) is a strictly convex program while  $B_k$  is positive definite, so it has an (unique) optimal solution;
- (iii)  $d$  is a solution of subproblem (QPs) if and only if it is a KKT point of subproblem (QPs).

Let  $d_0^k$  is an optimal solution of subproblem (QPs) at the  $k$ -th iteration, i.e., there exists a corresponding KKT multiplier vector  $\lambda^k = (\lambda_j^k, j \in I)$  such that

$$\begin{cases} g_0(x^k) + B_k d_0^k + \sum_{j \in I} \lambda_j^k g_j(x^k) = 0, \\ \bar{f}_j(x^k) + g_j(x^k)^T d_0^k \leq 0, \quad \lambda_j^k \geq 0, \quad \lambda_j^k (\bar{f}_j(x^k) + g_j(x^k)^T d_0^k) = 0, \quad \forall j \in I. \end{cases} \quad (4)$$

From (4) and the KKT condition for problem (NCP), the following lemma holds immediately.

**Lemma 2.1.**  *$x^k$  is a KKT point for problem (NCP) if and only if  $(d_0^k, \varphi_k) = (0, 0)$ .*

On one hand, in view of  $d = 0$  is a feasible solution and  $d_0^k$  is an optimal solution for subproblem (QPs), respectively, it follows that

$$g_0(x^k)^T d_0^k + \frac{1}{2}(d_0^k)^T B_k d_0^k \leq 0, \quad (5)$$

and if  $d_0^k \neq 0$ , together with the positive definiteness of  $B_k$ , (5) imply that

$$g_0(x^k)^T d_0^k \leq -\frac{1}{2}(d_0^k)^T B_k d_0^k < 0,$$

i.e.,  $d_0^k$  is a descent direction for problem (NCP) at  $x^k$ .

On the other hand,  $d_0^k$  may be not a feasible direction for problem (NCP) at the feasible iteration point  $x^k$ . Even when  $d_0^k$  is a feasible direction, a line search may not allow a full step of one to be taken in a neighborhood of an optimal solution and thus superlinear convergence may never take place. In order to get an improving direction and taking into account that  $x^k$  may be infeasible, we first propose a new SLE

$$V_k \begin{pmatrix} d \\ h \end{pmatrix} \triangleq \begin{pmatrix} B_k & N_k \\ N_k^T & -D^k \end{pmatrix} \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ -(\|d_0^k\| + \varphi_k^\sigma)\varpi \end{pmatrix} \quad (6)$$

to generate an updated direction  $\tilde{d}^k$ , where  $0 \in R^n$ ,  $\varpi = (1, 1, \dots, 1)^T \in R^m$ ,  $\sigma \in (0, 1)$  and

$$\begin{cases} N_k = (g_j(x^k), j \in I), & D^k = \text{diag}(D_j^k, j \in I), \\ D_j^k = |\bar{f}_j(x^k)|(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k| + \|d_0^k\|), & j \in I. \end{cases} \quad (7)$$

From (6) and (7), it follows that

$$g_j(x^k)^T \tilde{d}^k = -\|d_0^k\| - \varphi_k^\sigma, \quad \forall j \in I_k,$$

i.e.,  $\tilde{d}^k$  is a mere feasible direction.

The following lemma describes the solvability of SLE (6), its proof is similar to Lemma 2.2 in [15] and is omitted here.



**Lemma 2.2.** *Suppose that Assumptions 2.1 and 2.2 hold and  $B_k$  is positive definite. Then, the coefficient matrix  $V_k$  defined in (6) is nonsingular and (6) has a unique solution.*

Then, in order to yield an improving search direction at iteration point  $x^k$  (feasible or infeasible), we consider a convex combination of  $d_0^k$  and  $\tilde{d}^k$  as follows

$$q^k = (1 - \beta_k)d_0^k + \beta_k\tilde{d}^k, \quad (8)$$

where  $\beta_k$  is the maximum value of  $\beta \in [0, 1]$  that satisfies the following relationship

$$g_0(x^k)^T q^k = (1 - \beta_k)g_0(x^k)^T d_0^k + \beta_k g_0(x^k)^T \tilde{d}^k \leq \theta g_0(x^k)^T d_0^k + \varphi_k^\theta, \quad (9)$$

where the positive parameter  $\theta < \sigma$ . Taking into account that  $g_0(x^k)^T d_0^k \leq 0$ , it follows that  $\beta_k$  can be yielded by the following explicit formula

$$\beta_k = \begin{cases} \min\{1, \frac{(\theta-1)g_0(x^k)^T d_0^k + \varphi_k^\theta}{g_0(x^k)^T \tilde{d}^k - g_0(x^k)^T d_0^k}\}, & \text{if } g_0(x^k)^T \tilde{d}^k > g_0(x^k)^T d_0^k; \\ 1, & \text{if } g_0(x^k)^T \tilde{d}^k \leq g_0(x^k)^T d_0^k. \end{cases} \quad (10)$$

**Remark 2.2.** *From (9), it follows that the rate of increase of the objective function  $f_0$  at point  $x_k$  along direction  $q^k$  is bounded from upper by  $\theta g_0(x^k)^T d_0^k + \varphi_k^\theta$ , and it is just so ensuring the direction  $q^k$  possesses excellent convergence. In addition, the parameter  $\theta \in (0, \sigma)$  plays important roles in avoiding Maratos effect for analyzing the request (14) in the algorithm as well as forcing the iteration points always get into the feasible region after a finite number of iterations. These can be seen in the latter analysis, e.g., Lemma 3.1 and Theorems 4.2, 4.3 as well as 4.4.*

From the relationships of (8), (9) and (6), the following lemma can be proved easily.

**Lemma 2.3.** *Suppose that assumptions in Lemma 2.2 hold. Then*

*(i)  $g_0(x^k)^T q^k \leq -\frac{1}{2}\theta(d_0^k)^T B_k d_0^k + \varphi_k^\theta$ ; (ii)  $g_j(x^k)^T q^k \leq -\beta_k(\|d_0^k\| + \varphi_k^\sigma)$ ,  $\forall j \in I_k$ .*

From Lemma 2.3, it holds that  $q^k$  is an improving direction either for problem (NCP) or for the maximal violated constrained function  $\varphi(x)$ . In order to overcome the possibility of Maratos effect, a suitable higher-order correction direction must be introduced by an appropriate approach. Additionally, taking into account avoiding the strict complementarity condition, we introduce another SLE

$$V_k \begin{pmatrix} d \\ h \end{pmatrix} = \begin{pmatrix} 0 \\ -(\|d_0^k\|^\tau + \varphi_k^\sigma)\varpi - \tilde{F}(x^k + d_0^k) \end{pmatrix} \quad (11)$$

to yield a higher-order correction direction  $d_1^k$ , where  $\tau \in (2, 3)$  and

$$\tilde{F}(x^k + d_0^k) = (f_j(x^k + d_0^k) - f_j(x^k) - g_j(x^k)^T d_0^k, j \in I). \quad (12)$$

**Remark 2.3.** *The right-hand-side of (11) (in particular, the introducing of  $\tilde{F}(x^k + d_0^k)$ ) as well as  $d_1^k$  play an important role in the discussion of superlinear convergence in Section 4 without the strict complementarity, these can be found in the proofs of Lemma 4.1 and Theorem 4.2. In traditional analysis [1, 9, 16],  $d_1^k$  must satisfy  $\|d_1^k\| = O(\|d_0^k\|^2)$  that is called second-order correction. In this paper, the term  $\varphi_k$  is added, so the relation between  $d_1^k$  and  $d_0^k$  will be different from the traditional forms, see Lemma 4.1(i).*

Now, based on the analysis above, we can present our algorithm as follows.

**Algorithm 2.1.**

Parameters:  $\gamma, \eta, \varepsilon \in (0, 1)$ ,  $0 < \theta < \sigma < 1$ ,  $0 < \varrho < \sigma$ ,  $\xi, \zeta > 0$ ,  $\alpha \in (0, 0.5)$ ,  $\rho > 1$ ,  $\delta > 2$ ,  $\tau \in (2, 3)$ .

Data:  $x^0 \in R^n$ , a symmetric positive definite matrix  $B_0 \in R^{n \times n}$ . Set  $k := 0$ .

**Step 1.** Solve subproblem (QPs) to get a (unique) solution  $d_0^k$  and the corresponding KKT multiplier vector  $\lambda^k = (\lambda_j^k, j \in I)$ . If  $(d_0^k, \varphi_k) = 0$ , then  $x^k$  is a KKT point for problem (NCP) and stop.

**Step 2.** Compute the correction direction  $d_1^k$  by solving SLE (11) with a solution  $(d_1^k, h_1^k)$ , and let  $d^k = d_0^k + d_1^k$ . If

$$g_0(x^k)^T d_0^k \leq \zeta \min\{-||d_0^k||^\delta, -||d^k||^\delta\} + \xi \varphi_k^\varrho, \quad (13)$$

then go to Step 3; otherwise, go to Step 4.

**Step 3.** Let  $t = 1$ ,

(a) if

$$\begin{cases} f_0(x^k + td^k) \leq f_0(x^k) + \alpha t g_0(x^k)^T d_0^k + \rho(1 - \alpha)t \varphi_k^\theta, \\ f_j(x^k + td^k) \leq \varphi_k - \alpha t (||d_0^k||^\tau + \varphi_k^\sigma), \quad j \in I_k^+, \\ f_j(x^k + td^k) \leq 0, \quad j \in I_k^-, \end{cases} \quad (14)$$

is satisfied for the current value  $t$ , then let  $t_k = t$ , go to Step 6; otherwise, go to (b) below.

(b) Reset  $t := \frac{1}{2}t$ . If  $t < \varepsilon$ , then go to Step 4; otherwise, repeat part (a).

**Step 4.** Solve SLE (7) to get  $(\tilde{d}^k, \tilde{h}^k)$ , and compute  $\beta_k$  according to (10), then obtain the direction  $q^k$  by (8).

**Step 5.** Compute the steplength  $t_k$  which is the first number  $t$  in the sequence  $\{1, \eta, \eta^2, \dots\}$  that satisfies the following inequalities

$$f_0(x^k + tq^k) \leq f_0(x^k) + \gamma t g_0(x^k)^T q^k + \rho(1 - \gamma)t \varphi_k^\theta, \quad (15)$$

$$f_j(x^k + tq^k) \leq \varphi_k - \gamma t \beta_k (||d_0^k|| + \varphi_k^\sigma), \quad j \in I_k^+, \quad (16)$$

$$f_j(x^k + tq^k) \leq 0, \quad j \in I_k^-, \quad (17)$$

then let  $d^k = q^k$ .

**Step 6.** Compute a new symmetric positive definite matrix  $B_{k+1}$  by some suitable techniques, set  $x^{k+1} = x^k + t_k d^k$ ,  $k := k + 1$ , and go back to Step 1.

**Remark 2.4.** *From the mechanism of Algorithm 2.1, we know that when  $\varphi_k \neq 0$ , the value of the maximal violated constrained functions (namely  $\varphi_k$ ) is strictly decreasing, moreover, the amounts of decreasing have a close relation with  $||d_0^k|| + \varphi_k$ . Thus, the iteration points can approach the feasible region as soon as possible. In particular, they can always getting into the feasible set after a finite number of iterations.*

**Remark 2.5.** *It is known that the role of the request (13) is to restrict the increasing rate of the objective function  $f_0$  at point  $x^k$  along direction  $d_0^k$ . But, theoretically speaking, under the uniformly positive definite assumption (see Assumption 3.1 below) on the sequence  $\{B_k\}$  of matrices, the request (13) does not influence any theory analysis of Algorithm 2.1, this can be seen in the latter analysis. However, the request (13) still has some influence on the numerical effect of Algorithm 2.1. From the process of numerical experiments, it seems to be that, for small-scale problems, the numerical results of Algorithm 2.1 with the request (13) are better than the case of ignoring this request, and the case is inverse for middle-large-scale problems. In addition, the role of the exponents  $\delta > 2$  and  $\varrho < \sigma$  is to ensure, under suitable assumptions, that the request (13) is always satisfied when the iteration point  $x^k$  is close sufficiently to a KKT point, see Lemma 4.1(iii).*

**Remark 2.6.** *There are two cycles between Step 1 and Step 6, i.e., cycle I: Steps 1-2-3-6, and cycle II: Steps 1-2-4-5-6. Obviously, if the algorithm successfully perform cycle I, the cost of computation is relatively small. Fortunately, under suitable assumptions, we can prove that Algorithm 2.1 always performing cycle I after a finite number of iterations.*

The lemma given below indicates that the line search in Step 5 of Algorithm 2.1 is well defined.

**Lemma 2.4.** *Suppose that Assumptions 2.1 and 2.2 hold. Then, if Algorithm 2.1 does not stop at Step 1, i.e.,  $(d_0^k, \varphi_k) \neq (0, 0)$ , the line search (15)-(17) can be terminated after a finite number of computations.*

*Proof.* Suppose that  $(d_0^k, \varphi_k) \neq (0, 0)$  at the  $k$ -th iteration. From (10), it follows that  $\beta_k > 0$ .

(1) Analyze the inequality (15). From Taylor expansion and Lemma 2.3(i), it follows that

$$\begin{aligned} a_k(t) &\triangleq f_0(x^k + tq^k) - f_0(x^k) - \gamma tg_0(x^k)^T q^k - \rho(1 - \gamma)t\varphi_k^\theta \\ &= (1 - \gamma)tg_0(x^k)^T q^k - \rho(1 - \gamma)t\varphi_k^\theta + o(t\|q^k\|) \\ &\leq -\frac{1}{2}\theta(1 - \gamma)t(d_0^k)^T B_k d_0^k - (\rho - 1)(1 - \gamma)t\varphi_k^\theta + o(t\|q^k\|). \end{aligned} \quad (18)$$

This together with  $\theta, \gamma \in (0, 1)$ ,  $\rho > 1$  and  $(d_0^k)^T B_k d_0^k + \varphi_k^\theta > 0$  shows that  $a_k(t) \leq 0$  holds for  $t > 0$  sufficiently small.

(2) Analyze the inequalities (16).

(i) For  $j \in I_k^+ \cap I_k$ , it follows that  $f_j(x^k) = \varphi_k$ , expanding  $f_j(x^k + tq^k)$  around  $x^k$ , and combining Lemma 2.3(ii), for  $\gamma \in (0, 1)$  and  $t > 0$  sufficiently

small, we obtain

$$\begin{aligned}
f_j(x^k + tq^k) &= -\varphi_k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) \\
&= tg_j(x^k)^T q^k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t\|q^k\|) \\
&\leq -(1 - \gamma)t\beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t\|q^k\|) \\
&\leq 0.
\end{aligned}$$

(ii) For  $j \in I_k^+ \setminus I_k$ , we have  $f_j(x^k) - \varphi_k < 0$ , and furthermore,

$$\lim_{t \rightarrow 0^+} (f_j(x^k + tq^k) - \varphi_k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma)) = f_j(x^k) - \varphi_k < 0,$$

which implies the inequalities (16) hold for  $t > 0$  sufficiently small.

(3) Analyze the inequalities (17).

(i) For  $j \in I_k^- \cap I_k$ , expanding  $f_j(x^k + tq^k)$  at  $x^k$ , and taking into account Lemma 2.3(ii), for  $t > 0$  sufficiently small, it follows that

$$f_j(x^k + tq^k) = tg_j(x^k)^T q^k + o(t\|q^k\|) \leq -t\beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t\|q^k\|) \leq 0.$$

(ii) For  $j \in I_k^- \setminus I_k$ , we have  $f_j(x^k) < 0$ . So, from Assumption 2.1, it follows that  $f_j(x^k + tq^k) \leq 0$ , for  $t > 0$  sufficiently small.

Summarizing the analysis above, we conclude that there exists a  $\bar{t}_k > 0$  such that the line search (15)-(17) satisfies for all  $t \in (0, \bar{t}_k]$  and the given conclusion holds.  $\square$

At the end of this section, based on the line search conditions (14), (16) and (17), we can easily get the following lemma.

**Lemma 2.5.** *Suppose that Assumptions 2.1 and 2.2 hold. Then,*

*(i) for each  $k$ ,  $I_k^- \subseteq I_{k+1}^-$  holds, so, if there exists an iteration index  $k_0$  such that  $x^{k_0} \in X$ , i.e.,  $\varphi_{k_0} = 0$ , then  $x^k \in X$  for all  $k \geq k_0$ , and  $\{f_0(x^k)\}_{k \geq k_0}$  is*

decreasing;

(ii) if  $x^k \notin X$  and  $x^{k+1} \notin X$ , then  $\varphi_{k+1} \leq \varphi_k - t_k \max\{\alpha(\|d_0^k\|^\tau + \varphi_k^\sigma), \gamma\beta_k(\|d_0^k\| + \varphi_k^\sigma)\} < \varphi_k$ ;

(iii) for  $k$  large enough, the subsets  $I_k^-$  and  $I_k^+$  can be fixed, i.e.,  $I_k^- \equiv I^-$  and  $I_k^+ \equiv I^+$ .

**Remark 2.7.** From Lemma 2.5(i) and (ii), it follows that exactly one of the following two cases takes place:

Case A: There exists an iteration index  $k_0$  such that  $\varphi_{k_0} = 0$ , then  $\varphi_k \equiv 0$  for all  $k \geq k_0$ ;

Case B: For any  $k = 0, 1, 2, \dots$ ,  $\varphi_k > 0$  and  $\varphi_{k+1} < \varphi_k$ .

### 3. Global convergence

In this section, we will establish the global convergence of Algorithm 2.1. When Algorithm 2.1 stops at  $x^k$ , it follows that the iteration point  $x^k$  is a KKT point for problem (NCP) from Lemma 2.1 and Step 1 of Algorithm 2.1. Now, suppose that an infinite sequence  $\{x^k\}$  of iteration points is generated by Algorithm 2.1, and we will show that every accumulation point  $x^*$  of  $\{x^k\}$  is the KKT point for problem (NCP). For this purpose, the following assumption is necessary.

**Assumption 3.1.** The sequence  $\{B_k\}$  of matrices is uniformly positive definite, i.e., there exist two positive constants  $a$  and  $b$  such that

$$a\|d\|^2 \leq d^T B_k d \leq b\|d\|^2, \quad \forall d \in R^n, \quad \forall k. \quad (19)$$

Denote the active set for subproblem (QPs) by

$$J_k = \{j \in I : \bar{f}_j(x^k) + g_j(x^k)^T d_0^k = 0\}. \quad (20)$$

Suppose that  $x^*$  is a given accumulation point of  $\{x^k\}$ . In view of  $I_k^+$ ,  $I_k^-$ ,  $J_k$  all being subsets of the finite set  $I$  and Lemma 2.5(iii), we can assume without loss of generality that there exists an infinite index set  $K$  such that

$$x^k \rightarrow x^*, \quad I_k^- \equiv I^-, \quad I_k^+ \equiv I^+, \quad J_k \equiv J, \quad \varphi_k \rightarrow \varphi_* = \varphi(x^*), \quad \forall k \in K. \quad (21)$$

Based on the above conditions, we have the following lemmas.

**Lemma 3.1.** *Suppose that Assumptions 2.1, 2.2 and 3.1 hold. Then*

- (i) *sequence  $\{d_0^k\}_K \triangleq \{d_0^k : k \in K\}$  is bounded;*
- (ii) *there exists a constant  $c > 0$  such that  $\|V_k^{-1}\| \leq c$  for all  $k \in K$ , where  $V_k$  is the coefficient matrix defined by (6);*
- (iii) *sequences  $\{d_1^k\}_K$ ,  $\{\tilde{d}^k\}_K$ ,  $\{q^k\}_K$  and  $\{\tilde{h}^k\}_K$  are all bounded.*

*Proof.* (i) First, from  $\lim_{k \in K} x^k = x^*$  and the continuity of  $g_0(x)$ , there exists a constant  $\bar{c} > 0$  such that  $\|g_0(x^k)\| \leq \bar{c}$  holds for all  $k \in K$ . Then, from (5) and Assumption 3.1, it follows that  $-\bar{c}\|d_0^k\| + \frac{1}{2}a\|d_0^k\|^2 \leq 0$ ,  $\forall k \in K$ , which implies the boundedness of  $\{d_0^k\}_K$ .

(ii) The proof of conclusion (ii) is similar to Lemma 3.1 in [15], thus it is omitted here.

(iii) According to (6), (11) and parts (i) and (ii), it follows that  $\{d_1^k\}_K$ ,  $\{\tilde{d}^k\}_K$  and  $\{\tilde{h}^k\}_K$  are all bounded. Furthermore, from (8) and  $\beta_k \in [0, 1]$ , we obtain the boundedness of  $\{q^k\}_K$ .  $\square$

**Lemma 3.2.** *Suppose that assumptions stated in Lemma 3.1 hold. Then*

*$\lim_{k \in K} (d_0^k, \varphi_k) = (0, 0)$  and  $\lim_{k \in K} \tilde{d}^k = \lim_{k \in K} q^k = \lim_{k \in K} d_1^k = 0$ , where the index set  $K$  is defined by (21).*



*Proof.* We first show that  $\lim_{k \in K} (d_0^k, \varphi_k) = (0, 0)$ . Suppose by contradiction that there exists an infinite index set  $K' \subseteq K$  and a constant  $\tilde{\epsilon} > 0$  such that  $\|(d_0^k, \varphi_k)\| \geq \tilde{\epsilon}$ ,  $k \in K'$ . So, there exists an infinite set  $K'' \subseteq K'$  such that  $d_0^k \rightarrow d_0^*$ ,  $\varphi_k \rightarrow \varphi_*$ ,  $B_k \rightarrow B_*$ ,  $k \in K''$ . Thus, in view of (5), (19) and  $\theta \in (0, 1)$ , we have

$$(\theta - 1)g_0(x^k)^T d_0^k + \varphi_k^\theta \geq \frac{1}{2}(1 - \theta)a\|d_0^k\|^2 + \varphi_k^\theta \xrightarrow{k \in K''} \frac{1}{2}(1 - \theta)a\|d_0^*\|^2 + \varphi_*^\theta > 0,$$

which further shows that

$$\|d_0^k\|^2 + \varphi_k^\theta \xrightarrow{k \in K''} \|d_0^*\|^2 + \varphi_*^\theta > 0, \quad \|d_0^k\| + \varphi_k^\sigma \xrightarrow{k \in K''} \|d_0^*\| + \varphi_*^\sigma > 0.$$

This together with  $(d_0^k, \varphi_k) \neq 0$  and (10) implies that there exists a constant  $\bar{\epsilon} > 0$  such that

$$\beta_k \geq \bar{\epsilon}, \quad \|d_0^k\|^2 + \varphi_k^\theta \geq \bar{\epsilon}, \quad \|d_0^k\| + \varphi_k^\sigma \geq \bar{\epsilon}, \quad \forall k \in K''. \quad (22)$$

The rest proof is divided into two steps as follows.

**Step A.** Show that there exists a constant  $\bar{t} > 0$  such that the steplength  $t_k \geq \bar{t}$  for all  $k \in K''$ .

From the mechanism of Algorithm 2.1 we know that it is sufficient to prove that the inequalities (15)-(17) are satisfied for all  $k \in K''$  and  $t > 0$  small enough.

(1) Analysis the inequality (15). From the boundedness of  $\{d_0^k\}_K$  and (18), (19) as well as (22), we have

$$\begin{aligned} a_k(t) &\leq -(1 - \gamma)t(\frac{1}{2}\theta(d_0^k)^T B_k d_0^k + (\rho - 1)\varphi_k^\theta) + o(t) \\ &\leq -(1 - \gamma)t(\frac{1}{2}a\theta\|d_0^k\|^2 + (\rho - 1)\varphi_k^\theta) + o(t) \\ &\leq -(1 - \gamma)t \min\{\frac{1}{2}a\theta, \rho - 1\}(\|d_0^k\|^2 + \varphi_k^\theta) + o(t) \\ &\leq -(1 - \gamma)t\bar{\epsilon} \min\{\frac{1}{2}a\theta, \rho - 1\} + o(t) \leq 0 \end{aligned} \quad (23)$$

holds for  $t > 0$  small enough and all  $k \in K''$ .

(2) Analysis the inequalities (16).

(2-i) For  $j \in I^+$  and  $f_j(x^*) = \varphi_*$ , from (7) and (3), it follows that

$$D_j^k = |f_j(x^k) - \varphi_k|(|f_j(x^k) + g_j(x^k)^T d_0^k - \varphi_k| + \|d_0^k\|) \rightarrow 0, \quad k \in K''.$$

Therefore, from Taylor expansion, (8), (6) and the constraints of subproblem (QPs), for  $t > 0$  sufficiently small and all  $k \in K''$ , we obtain

$$\begin{aligned} & f_j(x^k + tq^k) - \varphi_k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) \\ &= f_j(x^k) + tg_j(x^k)^T q^k - \varphi_k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t) \\ &= f_j(x^k) - \varphi_k + t(1 - \beta_k)g_j(x^k)^T d_0^k + t\beta_k g_j(x^k)^T \tilde{d}^k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t) \\ &\leq (1 - t(1 - \beta_k))(f_j(x^k) - \varphi_k) - t\beta_k(\|d_0^k\| + \varphi_k^\sigma) + t\gamma\beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t) \\ &\leq -(1 - \gamma)t\beta_k(\|d_0^k\| + \varphi_k^\sigma) + o(t) \\ &\leq -\bar{\epsilon}^2(1 - \gamma)t + o(t) \leq 0. \end{aligned}$$

(2-ii) For  $j \in I^+$  and  $f_j(x^*) < \varphi_*$ , we have

$$\begin{aligned} f_j(x^k + tq^k) - \varphi_k + \gamma t \beta_k(\|d_0^k\| + \varphi_k^\sigma) &= f_j(x^k) - \varphi_k + O(t) \\ &\leq \frac{1}{2}(f_j(x^*) - \varphi_*) + O(t) \\ &\leq 0 \end{aligned}$$

holds for all  $k \in K''$  and  $t > 0$  sufficiently small.

(3) Analysis the inequalities (17). Similar to the analysis for the inequalities (16), one can show that (17) hold for all  $k \in K''$  and  $t > 0$  small enough.

Summarizing the analysis above, we conclude that there exists a  $\bar{t} > 0$  such that  $t_k \geq \bar{t}$  for all  $k \in K''$ .

**Step B.** Use  $t_k \geq \bar{t} > 0$  ( $k \in K''$ ) to bring a contradiction, and the discussion is divided into two cases.

Case I. Suppose that there exists an iteration index  $k_0$  such that  $\varphi_{k_0} = 0$ . Then  $\{f_0(x^k)\}_{k \geq k_0}$  is decreasing. Combining  $\lim_{k \in K''} f_0(x^k) = f_0(x^*)$ , it follows that  $\lim_{k \rightarrow +\infty} f_0(x^k) = f_0(x^*)$ . On the other hand, taking into account the first inequality of (14), (15), Lemma 2.3(i), (19) and (22) as well as  $\varphi_k \equiv 0$  ( $\forall k \geq k_0$ ), it follows that

$$\begin{aligned} f_0(x^{k+1}) &\leq f_0(x^k) + t_k \max\{\alpha g_0(x^k)^T d_0^k, \gamma g_0(x^k)^T q^k\} \\ &\leq f_0(x^k) + t_k \max\{-\frac{1}{2}\alpha a \|d_0^k\|^2, -\frac{1}{2}\theta\gamma a \|d_0^k\|^2\} \\ &\leq f_0(x^k) - \bar{t}\bar{\epsilon} \min\{\frac{1}{2}\alpha a, \frac{1}{2}\theta\gamma a\}, \quad \forall k \geq k_0, k \in K''. \end{aligned}$$

Thus, passing to the limit  $k \in K''$  and  $k \rightarrow +\infty$  in the inequality above, we can bring a contradiction.

Case II. Suppose that  $\varphi_k > 0$  for each  $k$ . Then  $\{\varphi_k\}_{k \geq 0}$  is decreasing. Combining  $\lim_{k \in K''} \varphi_k = \varphi_*$ , one knows that  $\lim_{k \rightarrow +\infty} \varphi_k = \varphi_*$ . On the other hand, taking into account the second inequality of (14), (16), Lemma 2.3(ii), (19) and (22) as well as  $\varphi_k > 0$  ( $\forall k \in K''$ ), it follows that

$$\begin{aligned} \varphi_{k+1} &\leq \varphi_k - t_k \max\{\alpha(\|d_0^k\|^\tau + \varphi_k^\sigma), \gamma\beta_k(\|d_0^k\| + \varphi_k^\sigma)\} \\ &\leq \varphi_k - \bar{t}\bar{\epsilon} \max\{\alpha, \gamma\bar{\epsilon}\}, \quad \forall k \in K'', k \text{ large enough.} \end{aligned}$$

Then, passing to the limit  $k \in K''$  and  $k \rightarrow +\infty$  in the above inequality, we can also bring a contradiction.

Up to now, we have finished the proof of  $\lim_{k \in K} (d_0^k, \varphi_k) = (0, 0)$ .

Finally, by Lemma 3.1(ii), (6), (11), (8) and  $\lim_{k \in K} (d_0^k, \varphi_k) = (0, 0)$ , it follows that  $\lim_{k \in K} \tilde{d}^k = \lim_{k \in K} q^k = \lim_{k \in K} d_1^k = 0$ .  $\square$

Now, we present the main result of this section.

**Theorem 3.1.** *Suppose that Assumptions 2.1, 2.2 and 3.1 hold. Then Algorithm 2.1 either stops at a KKT point  $x^k$  for problem (NCP) after a finite*

number of iterations or generates an infinite sequence  $\{x^k\}$  of points such that each accumulation point  $x^*$  (if it exists) of  $\{x^k\}$  is a KKT point for problem (NCP), furthermore, there exists an index set  $K$  such that  $\{(x^k, \lambda^k)\}_K$  converges to the KKT pair  $(x^*, \lambda^*)$  for problem (NCP).

*Proof.* Choose an infinite index set  $K$  such that (21) holds, and let matrix  $R_k = (g_j(x^k), j \in J)$ . In view of  $(x^k, d_0^k, \varphi_k) \rightarrow (x^*, 0, 0)$ ,  $k \in K$ , we can conclude that  $J \subseteq I(x^*) = \{j \in I : f_j(x^*) = 0\}$ , and this together with Assumption 2.2 shows that  $R_k^T R_k$  is nonsingular for  $k \in K$  large enough, since  $R_k \xrightarrow{K} R_* \triangleq (g_j(x^*), j \in J)$ . Again, from the KKT condition (4), we have

$$g_0(x^k) + B_k d_0^k + R_k \lambda_J^k = 0.$$

Thus, for  $k \in K$  sufficiently large, we have

$$\lambda_J^k = -(R_k^T R_k)^{-1} R_k^T (g_0(x^k) + B_k d_0^k) \rightarrow -(R_*^T R_*)^{-1} R_*^T g_0(x^*) \stackrel{def}{=} \lambda_J^*.$$

If we denote the multiplier vector  $\lambda^* = (\lambda_J^*, 0_{I \setminus J})$ , then  $\lim_{k \in K} \lambda^k = \lambda^*$ . Furthermore, passing to the limit  $k \in K$  and  $k \rightarrow \infty$  in (4), it follows that

$$g_0(x^*) + N_* \lambda^* = 0, f_j(x^*) \leq 0, \lambda_j^* \geq 0, f_j(x^*) \lambda_j^* = 0, j \in I,$$

which shows that  $(x^*, \lambda^*)$  is a KKT pair for the problem (NCP).  $\square$

**Remark 3.1.** *The global convergence Theorem 3.1 shows that if the sequence  $x^k$  generated by the proposed Algorithm 2.1 possesses a limit point  $x^*$ , then  $x^*$  is a KKT point for problem (NCP).*

#### 4. Superlinear convergence

In this section, under some suitable assumptions, we first prove that Algorithm 2.1 possesses strong convergence, and the iteration points always enter

into the feasible set  $X$  after a finite number of iterations. Subsequently, the superlinear rate of convergence is established without the strict complementarity. For these purposes, we further make the following assumption.

**Assumption 4.1.** (i) The functions  $f_j(x)$  ( $j \in \{0\} \cup I$ ) are all second-order continuously differentiable.

(ii) The sequence  $\{x^k\}$  generated by Algorithm 2.1 is bounded, and possesses an accumulation point  $x^*$ , such that the KKT pair  $(x^*, \lambda^*)$  satisfies the strong second-order sufficient conditions (SSOSC for short), i.e.,

$$d^T \nabla_{xx}^2 L(x^*, \lambda^*) d > 0, \quad \forall d \in R^n, \quad d \neq 0, \quad g_j(x^*)^T d = 0, \quad j \in I_*^+,$$

where  $L(x, \lambda) = f_0(x) + \sum_{j \in I} \lambda_j f_j(x)$ ,  $I_*^+ = \{j \in I : \lambda_j^* > 0\}$ .

**Remark 4.1.** On the one hand, in some previously proposed SQP-type algorithms [1, 9], to get the superlinear convergence of the proposed algorithm, one has to ensure relation  $J_k \equiv I(x^*)$  holds for  $k$  large enough. And the strict complementarity (i.e.,  $I_*^+ = I(x^*)$ ) is an very important condition for ensuring  $J_k \equiv I(x^*)$  holds. On the other hand, it is well-known that the SSOSC is equivalent to the second-order sufficient conditions (SOSC for short) under the condition of the strict complementarity, however, this condition is hard to verify in practise, and the positive space (i.e., the critical directions set) of the SOSC is smaller than the SSOSC's. In this paper, in order to obtain the strong and superlinear convergence of Algorithm 2.1, combining a slight strong assumption SSOSC, we can still avoid the Maratos effect under  $I_*^+ \subseteq J_k \subseteq I(x^*)$  (see Lemma 4.1, Theorem 4.2).

First, under the stated assumptions, we have the following theorem.

**Theorem 4.1.** *Suppose that Assumptions 2.2, 3.1 and 4.1 hold. Then*

$$(i) \lim_{k \rightarrow +\infty} d_0^k = \lim_{k \rightarrow +\infty} d_1^k = \lim_{k \rightarrow +\infty} \tilde{d}^k = \lim_{k \rightarrow +\infty} q^k = \lim_{k \rightarrow +\infty} d^k = 0 \text{ and } \lim_{k \rightarrow +\infty} \|x^{k+1} - x^k\| = 0;$$

(ii)  $\lim_{k \rightarrow +\infty} x^k = x^*$ , i.e., Algorithm 2.1 is said to be strongly convergent in this sense;

$$(iii) \lim_{k \rightarrow +\infty} \varphi_k = \varphi(x^*) = 0, \quad \lim_{k \rightarrow +\infty} \lambda^k = \lambda^*.$$

*Proof.* (i) Since  $\{x^k\}$  is bounded, from Lemma 3.2, it follows that any subsequence of  $\{(d_0^k, d_1^k, \tilde{d}^k, q^k)\}$  must possess an accumulation point  $(0, 0, 0, 0) \in R^{4n}$ , which implies  $\lim_{k \rightarrow +\infty} (d_0^k, d_1^k, \tilde{d}^k, q^k) = (0, 0, 0, 0)$ . Furthermore, from the mechanism of Algorithm 2.1, we obtain

$$\lim_{k \rightarrow +\infty} d^k = 0, \quad \lim_{k \rightarrow +\infty} \|x^{k+1} - x^k\| = \lim_{k \rightarrow +\infty} \|t_k d^k\| = 0.$$

(ii) First, under Assumptions 2.2 and 4.1, by Proposition 4.1 in [1], it is known that the given limit point  $x^*$  is an isolated KKT point of problem (NCP). Therefore, we know from Theorem 3.1 that  $x^*$  is an isolated limit point of  $\{x^k\}$ , this combining  $\|x^{k+1} - x^k\| \rightarrow 0$ , it follows that  $x^k \rightarrow x^*$  (The details can be found in [14]).

(iii) In view of Theorem 3.1, we know that the given accumulation point  $x^*$  is a KKT point for problem (NCP), thus it follows  $\varphi(x^*) = 0$ . Furthermore, combining the monotone property and boundedness of  $\{\varphi_k\}$ , we have  $\lim_{k \rightarrow +\infty} \varphi_k = \varphi(x^*) = 0$ . Moreover, from the proof of Theorem 3.1 and parts (i) and (ii), one can conclude that each accumulation point of sequence  $\{\lambda_k\}$  is a KKT multiplier associated with  $x^*$ , this together with the uniqueness of the KKT multiplier implies that  $\lim_{k \rightarrow +\infty} \lambda^k = \lambda^*$ .  $\square$

The relations established in the following lemma are very important in

the subsequent discussion.

**Lemma 4.1.** *Suppose that Assumptions 2.2, 3.1 and 4.1 all hold. Then*

(i)  $\|d_1^k\| = O(\|d_0^k\|^2) + O(\varphi_k^\sigma)$ ,  $\|d_1^k\|^2 = O(\|d_0^k\|^4) + o(\varphi_k^\sigma)$ ,  $\|h_1^k\| = O(\|d_0^k\|^2) + O(\varphi_k^\sigma)$ ;

(ii)  $I_*^+ \subseteq J_k \subseteq I(x^*)$  for  $k$  large enough;

(iii) the relationship (13) is satisfied for  $k$  large enough.

*Proof.* (i) In view of  $\tilde{F}(x^k + d_0^k) = O(\|d_0^k\|^2)$ , the proof is elementary from (11) and Lemma 3.1(ii) as well as Theorem 4.1.

(ii) For  $j \notin I(x^*)$ , i.e.,  $f_j(x^*) < 0$ . Since  $\lim_{k \rightarrow +\infty} (x^k, d_0^k) = (x^*, 0)$ , there exists a constant  $\bar{\xi} > 0$  such that  $\bar{f}_j(x^k) = f_j(x^k) \leq -\bar{\xi} < 0$  holds for  $k$  large enough. It then follows that  $\bar{f}_j(x^k) + g_j(x^k)^T d_0^k \leq -\frac{1}{2}\bar{\xi} < 0$  for  $k$  large enough, which shows that  $j \notin J_k$ , thus  $J_k \subseteq I(x^*)$ . Furthermore, it follows from Theorem 4.1 that  $\lim_{k \rightarrow +\infty} \lambda_{I_*^+}^k = \lambda_{I_*^+}^* > 0$ , so,  $\lambda_{I_*^+}^k > 0$  and  $I_*^+ \subseteq J_k$  hold for  $k$  sufficiently large.

(iii) From (5) and Assumption 3.1, we only need to show that

$$-\frac{a}{2}\|d_0^k\|^2 \leq \zeta \min\{-\|d_0^k\|^\delta, -\|d^k\|^\delta\} + \xi\varphi_k^g$$

holds for  $k$  large enough.

First, in view of  $\delta > 2$  and Lemma 3.2, it follows that

$$-\frac{a}{2}\|d_0^k\|^2 \leq -\zeta\|d_0^k\|^\delta \leq -\zeta\|d^k\|^\delta + \xi\varphi_k^g, \quad (24)$$

for  $k$  sufficiently large and  $\zeta, a > 0$ .

Second, in view of  $\|d^k\| = \|d_0^k + d_1^k\| \leq \|d_0^k\| + \|d_1^k\|$ ,  $\delta > 2$ , part (i) and Lemma 3.2, we obtain

$$\|d^k\|^\delta \leq (\|d_0^k\| + O(\|d_0^k\|^2) + O(\varphi_k^\sigma))^\delta = \|d_0^k\|^\delta + o(\|d_0^k\|^2) + o(\varphi_k^\sigma)$$

for  $k$  large enough. This together with  $\delta > 2$  and  $\varrho < \sigma$  implies that

$$\begin{aligned} -\zeta \|d^k\|^\delta + \xi \varphi_k^\varrho + \frac{a}{2} \|d_0^k\|^2 &\geq -\zeta \|d_0^k\|^\delta + \frac{a}{2} \|d_0^k\|^2 + o(\|d_0^k\|^2) + \xi \varphi_k^\varrho + o(\varphi_k^\sigma) \\ &= \frac{a}{2} \|d_0^k\|^2 + o(\|d_0^k\|^2) + \xi \varphi_k^\varrho + o(\varphi_k^\sigma) \\ &\geq 0 \end{aligned}$$

holds for  $k$  large enough, i.e.,

$$-\frac{a}{2} \|d_0^k\|^2 \leq -\zeta \|d^k\|^\delta + \xi \varphi_k^\varrho, \quad \text{for } k \text{ large enough.} \quad (25)$$

Finally, combining (24), (25) and (5), the conclusion (iii) holds for  $k$  large enough.  $\square$

To ensure that the steplength  $t_k \equiv 1$  for  $k$  large enough without the strict complementary assumption, an additional assumption as follows is necessary.

**Assumption 4.2.** *Suppose that the KKT pair  $(x^*, \lambda^*)$  and the matrix  $B_k$  satisfy*

$$\|(\nabla_{xx}^2 L(x^*, \lambda^*) - B_k)d_0^k\| = o(\|d_0^k\|).$$

**Remark 4.2.** *According to Theorem 4.4, it holds that*

$$\textbf{Assumption 5} \iff \|(\nabla_{xx}^2 L(x^k, \lambda^k) - B_k)d_0^k\| = o(\|d_0^k\|).$$

**Theorem 4.2.** *Suppose that Assumptions 2.2, 3.1, 4.1 and 4.2 are all satisfied. Then, the inequality (14) in Step 3 is always satisfied for  $t = 1$  and sufficiently large  $k$ . Therefore, Step 4 and Step 5 are no longer performed in Algorithm 2.1, and Algorithm 2.1 always performs cycle I.*

*Proof.* First of all, in view of Lemma 4.1(iii), it is sufficient to prove that the inequality (14) holds for  $t = 1$  and sufficiently large  $k$ .



Discuss the second group of inequalities and the last group of inequalities of (14).

For  $j \notin I(x^*)$ , i.e.,  $f_j(x^*) < 0$ , in view of  $(x^k, d_0^k, d_1^k, \varphi_k) \rightarrow (x^*, 0, 0, 0)$  ( $k \rightarrow +\infty$ ), we obtain  $d^k = d_0^k + d_1^k \rightarrow 0$  ( $k \rightarrow +\infty$ ), thus, we conclude that the second group of inequalities and the last group of inequalities of (14) are both satisfied for  $t = 1$  and  $k$  large enough.

For  $j \in I(x^*)$ , since  $\lim_{k \rightarrow +\infty} f_j(x^k) = f_j(x^*) = 0$  and  $\lim_{k \rightarrow +\infty} \varphi_k = 0$  as well as (3) and (7), it follows that

$$\bar{f}_j(x^k) \rightarrow 0, \quad D_j^k \rightarrow 0, \quad D_j^k = o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) + o(\|d_0^k\|).$$

On the other hand, from (11) and Lemma 4.1(i), we have

$$\begin{aligned} g_j(x^k)^T d_1^k &= -\|d_0^k\|^\tau - \varphi_k^\sigma - f_j(x^k + d_0^k) + f_j(x^k) + g_j(x^k)^T d_0^k + D_j^k h_{1j}^k \\ &= -\|d_0^k\|^\tau - \varphi_k^\sigma - f_j(x^k + d_0^k) + f_j(x^k) + g_j(x^k)^T d_0^k \\ &\quad + o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) + O(\|d_0^k\|^3) + o(\varphi_k^\sigma). \end{aligned} \tag{26}$$

Then, from Taylor expansion, Lemma 4.1(i), (26) and  $\tau \in (2, 3)$ , we obtain

$$\begin{aligned} f_j(x^k + d^k) &= f_j(x^k + d_0^k) + g_j(x^k + d_0^k)^T d_1^k + O(\|d_1^k\|^2) \\ &= f_j(x^k + d_0^k) + g_j(x^k)^T d_1^k + O(\|d_0^k\|^3) + o(\varphi_k^\sigma) \\ &= -\|d_0^k\|^\tau - \varphi_k^\sigma + f_j(x^k) + g_j(x^k)^T d_0^k + o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) \\ &\quad + O(\|d_0^k\|^3) + o(\varphi_k^\sigma), \end{aligned} \tag{27}$$

which further implies that

$$f_j(x^k + d^k) = \begin{cases} -||d_0^k||^\tau - \varphi_k^\sigma - |\bar{f}_j(x^k) + g_j(x^k)^T d_0^k| + o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) \\ \quad + O(||d_0^k||^3) + o(\varphi_k^\sigma), & \text{if } j \in I(x^*) \cap I^-; \\ -||d_0^k||^\tau - \varphi_k^\sigma + \varphi_k - |\bar{f}_j(x^k) + g_j(x^k)^T d_0^k| + o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) \\ \quad + O(||d_0^k||^3) + o(\varphi_k^\sigma), & \text{if } j \in I(x^*) \cap I^+. \end{cases} \quad (28)$$

This shows that  $f_j(x^k + d^k) \leq 0$ , for  $j \in I(x^*) \cap I^-$  and  $k$  large enough.

For  $j \in I(x^*) \cap I^+$ , from (28) and  $\tau \in (2, 3)$  as well as  $\alpha \in (0, \frac{1}{2})$ , it follows that

$$\begin{aligned} f_j(x^k + d^k) - \varphi_k + \alpha(||d_0^k||^\tau + \varphi_k^\sigma) \\ = -(1 - \alpha)(||d_0^k||^\tau + \varphi_k^\sigma) - |\bar{f}_j(x^k) + g_j(x^k)^T d_0^k| \\ + o(|\bar{f}_j(x^k) + g_j(x^k)^T d_0^k|) + o(\varphi_k^\sigma) + O(||d_0^k||^3) \\ \leq 0. \end{aligned} \quad (29)$$

Thus, summarizing the analysis above, we have proved that the inequality of (14) except the first one is satisfied for  $t = 1$  and  $k$  large enough.

Finally, we will show that the first inequality of (14) holds for  $t = 1$  and  $k$  large enough. From Taylor expansion and Lemma 4.1(i), we have

$$\begin{aligned} \Delta_k &\triangleq f_0(x^k + d^k) - f_0(x^k) - \alpha g_0(x^k)^T d_0^k - \rho(1 - \alpha)\varphi_k^\theta \\ &= g_0(x^k)^T d^k + \frac{1}{2}(d^k)^T \nabla_{xx}^2 f_0(x^k) d^k - \alpha g_0(x^k)^T d_0^k - \rho(1 - \alpha)\varphi_k^\theta + o(||d^k||^2) \\ &= g_0(x^k)^T (d_0^k + d_1^k) + \frac{1}{2}(d_0^k)^T \nabla_{xx}^2 f_0(x^k) d_0^k - \alpha g_0(x^k)^T d_0^k - \rho(1 - \alpha)\varphi_k^\theta \\ &\quad + o(||d_0^k||^2) + o(\varphi_k^\sigma). \end{aligned} \quad (30)$$

Then, from the KKT conditions (4) and Lemma 4.1(i), we have

$$g_0(x^k)^T d_0^k = -(d_0^k)^T B_k d_0^k - \sum_{j \in J_k} \lambda_j^k g_j(x^k)^T d_0^k, \quad (31)$$

$$g_0(x^k)^T(d_0^k + d_1^k) = -(d_0^k)^T B_k d_0^k - \sum_{j \in J_k} \lambda_j^k g_j(x^k)^T(d_0^k + d_1^k) + o(\|d_0^k\|^2) + o(\varphi_k^\sigma). \quad (32)$$

For  $j \in J_k \subseteq I(x^*)$ , it follows that  $\bar{f}_j(x^k) + \nabla f_j(x^k)^T d_0^k = 0$ , which together with (28) and  $\varphi_k = o(\varphi_k^\sigma)$  implies

$$f_j(x^k + d^k) = -\|d_0^k\|^\tau - \varphi_k^\sigma + o(\|d_0^k\|^2) + o(\varphi_k^\sigma), \quad j \in J_k. \quad (33)$$

Again, from Taylor expansion and Lemma 4.1(i) as well as  $\varphi_k = o(\varphi_k^\sigma)$ , we get

$$\begin{aligned} f_j(x^k + d^k) &= f_j(x^k) + g_j(x^k)^T(d_0^k + d_1^k) + \frac{1}{2}(d_0^k)^T \nabla_{xx}^2 f_j(x^k) d_0^k + o(\|d_0^k\|^2) + o(\varphi_k^\sigma) \\ &= \bar{f}_j(x^k) + g_j(x^k)^T(d_0^k + d_1^k) + \frac{1}{2}(d_0^k)^T \nabla_{xx}^2 f_j(x^k) d_0^k + o(\|d_0^k\|^2) + o(\varphi_k^\sigma). \end{aligned} \quad (34)$$

Combining (33) and (34), it follows that

$$\begin{aligned} - \sum_{j \in J_k} \lambda_j^k g_j(x^k)^T(d_0^k + d_1^k) &= \sum_{j \in J_k} \lambda_j^k \bar{f}_j(x^k) + \frac{1}{2} \sum_{j \in J_k} \lambda_j^k (d_0^k)^T \nabla_{xx}^2 f_j(x^k) d_0^k \\ &\quad + o(\|d_0^k\|^2) + O(\varphi_k^\sigma). \end{aligned} \quad (35)$$

Thus, substituting (35) into (32), we have

$$\begin{aligned} g_0(x^k)^T(d_0^k + d_1^k) &= -(d_0^k)^T B_k d_0^k + \sum_{j \in J_k} \lambda_j^k \bar{f}_j(x^k) + \frac{1}{2}(d_0^k)^T (\nabla_{xx}^2 L(x^k, \lambda_k) \\ &\quad - \nabla_{xx}^2 f_0(x^k)) d_0^k + o(\|d_0^k\|^2) + O(\varphi_k^\sigma). \end{aligned} \quad (36)$$

In addition, in view of  $g_j(x^k)^T d_0^k = -\bar{f}_j(x^k)$ ,  $j \in J_k$ , from (31), it follows that

$$g_0(x^k)^T d_0^k = -(d_0^k)^T B_k d_0^k + \sum_{j \in J_k} \lambda_j^k \bar{f}_j(x^k). \quad (37)$$

Now, substituting (36) and (37) into (30), we have

$$\begin{aligned}
\Delta_k &= -(d_0^k)^T B_k d_0^k + \sum_{j \in J_k} \lambda_j^k \bar{f}_j(x^k) + \frac{1}{2} (d_0^k)^T \nabla_{xx}^2 L(x^k, \lambda_k) d_0^k - \alpha g_0(x^k)^T d_0^k \\
&\quad - \rho(1 - \alpha) \varphi_k^\theta + o(\|d_0^k\|^2) + O(\varphi_k^\sigma) \\
&= (\alpha - \frac{1}{2}) (d_0^k)^T B_k d_0^k + (1 - \alpha) \sum_{j \in J_k} \lambda_j^k \bar{f}_j(x^k) + \frac{1}{2} (d_0^k)^T (\nabla_{xx}^2 L(x^k, \lambda_k) \\
&\quad - B_k) d_0^k - \rho(1 - \alpha) \varphi_k^\theta + o(\|d_0^k\|^2) + O(\varphi_k^\sigma).
\end{aligned}$$

This together with Assumptions 3.1, 4.2,  $\lambda_j^k \bar{f}_j(x^k) \leq 0$  and  $\alpha \in (0, \frac{1}{2})$  as well as  $\theta < \sigma$  shows that

$$\Delta_k \leq (\alpha - \frac{1}{2}) a \|d_0^k\|^2 + o(\|d_0^k\|^2) - \rho(1 - \alpha) \varphi_k^\theta + o(\varphi_k^\theta) \leq 0$$

holds for  $k$  large enough. Hence, the first inequality of (14) holds for  $t = 1$  and  $k$  large enough. The whole proof is completed.  $\square$

**Remark 4.3.** *In order to overcome the Maratos effect, under some suitable assumptions, some norm relations of directions and a weaker set relation corresponding to the strict complementarity (i.e.,  $I_*^+ = I(x^*)$ ) are given in Lemma 4.1 firstly. Then, by these results, we obtain the Theorem 4.2, i.e., the inequality (14) is always satisfied for  $t = 1$  and  $k$  large enough (i.e., very close to the solution of the problem). So, the Maratos effect can be overcome in our paper.*

According to Theorem 4.2 and its proof for case of  $j \notin I(x^*)$  as well as relationship (28), the following lemma holds immediately.

**Theorem 4.3.** *Under the assumptions stated in Theorem 4.2, we have  $\varphi_{k+1} \equiv 0$  after a finite number of iterations, i.e.,  $x^{k+1} \in X$  for  $k$  large enough.*

At the end of this section, based on Theorems 4.2 and 4.3 as well as Lemma 4.1, using Theorem 3.1.3 in [14], we have the superlinear convergence of Algorithm 2.1 immediately as follows.

**Theorem 4.4.** *Suppose that Assumptions 2.2, 3.1, 4.1 and 4.2 are all satisfied. Then,  $\|x^{k+1} - x^*\| = o(\|x^k - x^*\|)$ , i.e., Algorithm 2.1 is superlinearly convergent.*

## 5. Numerical experiments

In this section, in order to illustrate the computational efficiency of Algorithm 2.1, some preliminary numerical results are reported, and the computing results show that Algorithm 2.1 is effective. The algorithm was implemented by using Matlab 7.5 on Windows XP platform, and on a PC with 1.99 GHZ CPU. The approximation Hessian matrix  $B_k$  is updated by the BFGS formula described in [17].

During the numerical experiments, the parameters are selected as follows:

$$\begin{cases} \gamma = \eta = 0.5, \theta = \varrho = 0.4, \sigma = 0.6, \xi = 1, \zeta = 0.2, \\ \alpha = 0.3, \rho = 1.5, \delta = 3, \tau = 2.5, \varepsilon = 0.5^3. \end{cases}$$

We test some problems which are taken from [18, 19]. In addition, we further test Svanberg problems in different dimensions and with different initial points, which are taken from [20]. Execution is terminated if the norm of  $d_0^k$  is less than a given constant  $\epsilon > 0$  and  $\varphi_k = 0$ . The columns of the following tables have the following meanings:

Prob: the number of the test problem in [18, 19];

$n/m$ : the number of variables/inequality constraints of the problem;

Code: the name of the algorithm;

NF0: the number of objective function evaluations;

NIO/NII: the number of iterations out of/within the feasible set;

NI: the total number of iterations, i.e.,  $NI=NIO+NII$ ;

NF: the number of all constraint functions evaluations;

FV: the objective function value at the final iteration point;

CPU: the CPU time (second).

Finally, an “—” in the following tables indicates that the corresponding information is not given in the corresponding references.

**Experiment 1** (*for small-scale problems*). For this part, in order to show the computational efficiency of Algorithm 2.1 (denoted by ALG 2.1), we test some small-scale problems and compare ALG 2.1 with some other algorithms, and the numerical results are given in Tables 1-4.

In Tables 1 and 2, ALG 2.1 is compared with ALGO [8] and SNQP [15] for the same test problems, the stopping criterion threshold  $\epsilon$  and initial iteration points are the same as that reported in [8] and [15], respectively. From the viewpoint of the numbers of NIO, it follows that ALG 2.1 can always enter into  $X$  after relatively small iterations. Furthermore, from the viewpoint of the numbers of NIO, NII and FV, the results show that ALG 2.1 is obviously better than ALGO for most of test problems. The performance of ALG 2.1 in terms of NII is better than SNQP except problems 33 and 76.

Tables 3 and 4 gives the compared numerical results for ALG 2.1 and ALG 3.1 as well as ALG 3.2 [21]. The test problems and stopping criterion threshold are the same as in [21]. The numerical results in Tables 3 and 4 show that ALG 2.1 can always enter into  $X$  after small iterations, and ALG

2.1 is more better than ALG 3.1 and ALG 3.2 for the test problems.

Table 1. *Numerical results for Experiment 1-I*

Prob	$n/m$	Initial point	Code	NIO	NII	NF0	NF	FV	CPU
012	2/1	$(6, 6)^T$	ALG 2.1	17	3	21	41	$-3.0000000E + 01$	0.06
			ALGO	25	28	29	57	$-3.0000000E + 01$	—
			SNQP	7	12	12	29	$-2.9999999E + 01$	—
029	3/1	$(-4, -4, -4)^T$	ALG 2.1	3	9	13	46	$-2.2627417E + 01$	0.05
			ALGO	1	11	14	27	$-2.2627417E + 01$	—
			SNQP	1	12	17	42	$-2.2627416E + 01$	—
031	3/7	$(2, 4, 7)^T$	ALG 2.1	1	16	18	309	$6.0000000E + 00$	0.06
			ALGO	4	20	23	43	$6.0000000E + 00$	—
			SNQP	1	19	20	52	$6.0000089E + 00$	—
033	3/6	$(2, 4, 6)^T$	ALG 2.1	1	9	11	134	$-4.5857864E + 00$	0.05
			ALGO	2	16	17	67	$-4.5857863E + 00$	—
		$(1, 4, 6)^T$	ALG 2.1	1	44	46	570	$-4.5857864E + 00$	0.33
			SNQP	2	21	23	116	$-4.5857290E + 00$	—
034	3/8	$(2, 2, 2)^T$	ALG 2.1	5	10	16	166	$-8.3403245E - 01$	0.06
			ALGO	8	26	68	198	$-8.3403244E - 01$	—
035	3/4	$(1, 2, 3)^T$	ALG 2.1	1	6	8	67	$1.1111111E - 01$	0.03
			ALGO	8	11	12	0	$-3.4500000E + 00$	—
			SNQP	4	9	9	0	$1.1111111E - 01$	—
043	4/3	$(-10, 2, -8, 5)^T$	ALG 2.1	9	5	15	95	$-4.4000000E + 01$	0.06
			ALGO	23	26	27	163	$-4.4000000E + 01$	—
		$(0, 2, 2, 4)^T$	ALG 2.1	7	9	17	135	$-4.4000000E + 01$	0.08
			SNQP	1	11	11	69	$-4.3999999E + 01$	—
044	4/10	$(-20, -20, -20, -20)^T$	ALG 2.1	4	10	15	296	$-1.5000000E + 01$	0.08
			ALGO	7	15	18	0	$-1.5000000E + 01$	—
066	3/8	$(0, 0, 100)^T$	ALG 2.1	10	54	65	1067	$5.1816327E - 01$	0.48
			ALGO	34	39	40	161	$5.1816327E - 01$	—
076	4/7	$(1, 2, 3, 4)^T$	ALG 2.1	5	16	22	345	$-4.6818182E + 00$	0.11
			ALGO	6	14	15	0	$-4.6818182E + 00$	—
			SNQP	2	14	14	0	$-4.6818171E + 00$	—
100	7/4	$(0, 3, -3, 3, 0, 1, 0)^T$	ALG 2.1	18	39	58	861	$6.8256637E + 02$	0.61
			ALGO	4	33	47	363	$6.8063006E + 02$	—

Table 2. *Numerical results for Experiment 1-I-continued*

Prob	$n/m$	Initial point	Code	NIO	NII	NF0	NF	FV	CPU
113	10/8	(4, 10, 10, 2, 0, 11, 4, 0, 12, 10) <sup>T</sup>	ALG 2.1	12	4	17	378	2.4306209E + 01	0.14
			ALGO	6	17	21	205	2.4306209E + 01	—
		(0, 2, 9, 5, 0, 1, 9, 8, -10, 10) <sup>T</sup>	ALG 2.1	9	7	17	355	2.4306585E + 01	0.17
			SNQP	9	29	29	258	2.4306211E + 01	—
264	4/3	(8, -5, 6, -4) <sup>T</sup>	ALG 2.1	18	5	24	142	-4.3987578E + 01	0.13
			ALGO	19	26	27	161	-4.3999999E + 01	—
		(0, 0, 0, 10) <sup>T</sup>	ALG 2.1	17	5	23	146	-4.3987578E + 01	0.16
			SNQP	4	16	17	122	-4.4113405E + 01	—

Table 3. *Numerical results for Experiment 1-II*

Prob	$n/m$	Initial point	Code	NIO+NII	NF0+NF	NDF0+NG	CPU
Rosen/Suzuki-1	4/3	(0, 0, 0, 0) <sup>T</sup>	ALG A	0+17	18+117	18+54	0.09
			ALG 3.1	76	1185	308	2.88
			ALG 3.2	77	1225	312	3.04
Rosen/Suzuki-2	4/3	(2, 4, 8, 1) <sup>T</sup>	ALG A	9+10	20+120	20+60	0.09
			ALG 3.1	68	1034	276	2.36
			ALG 3.2	55	793	224	2.06
Wong Problem-1	7/4	(1, 2, 0, 4, 0, 1, 1) <sup>T</sup>	ALG A	0+24	25+228	25+100	0.16
			ALG 3.1	157	17756	790	21.48
			ALG 3.2	157	17759	790	21.54
Wong Problem-2	7/4	(3, 3, 0, 5, 1, 3, 0) <sup>T</sup>	ALG A	9+48	58+624	58+232	0.38
			ALG 3.1	171	18677	860	22.48
			ALG 3.2	151	16921	760	20.88
Quadratic Problem-1	2/2	(-0.3, 0.0) <sup>T</sup>	ALG A	0+7	8+38	8+16	0.03
			ALG 3.1	48	292	147	0.76
			ALG 3.2	49	301	150	0.76
Quadratic Problem-2	2/2	(2.2, 1.6) <sup>T</sup>	ALG A	6+4	11+42	11+22	0.06
			ALG 3.1	50	314	153	0.72
			ALG 3.2	43	286	132	0.68



Table 4. *Numerical results for Experiment 1-II-continued*

Prob	$n/m$	Initial point	Code	NIO+NII	NF0+NF	NDF0+NG	CPU
TFI1 Problem-1	3/1	$(-10, 0, 0)^T$	ALG A	0+12	13+32	13+13	0.05
			ALG 3.1	35	2663	792	1.46
			ALG 3.2	35	2663	792	1.46
TFI1 Problem-2	3/1	$(1, 1, 1)^T$	ALG A	1+7	9+17	9+9	0.05
			ALG 3.1	23	2301	528	0.98
			ALG 3.2	22	2289	506	0.96
TFI2 Problem-1	3/1	$(2, 2, 2)^T$	ALG A	0+6	7+23	7+9	0.04
			ALG 3.1	30	1343	682	1.04
			ALG 3.2	30	1343	682	1.04
TFI2 Problem-2	3/1	$(0, 0, 0)^T$	ALG A	4+7	12+25	12+18	0.06
			ALG 3.1	48	2129	1078	1.58
			ALG 3.2	48	2135	1078	1.60

**Experiment 2** (*for middle-large-scale problems*). Considering the all of tested problems above are all relatively small, we further test the Svanberg problems [20] problems, some of them are larger and therefore interesting. The experiment results are given in Tables 5-6.

In Table 5, the performance of ALG 2.1 is compared with SNQP, ALGO, FSLE [22]. The initial iteration points and the stopping criterion threshold are the same as that reported in [22]. From the results in Table 5, in viewpoint of NII and NF0, it follows that algorithm ALG 2.1 performs better than FSLE, SNQP and ALGO in most cases for problems Svanberg.

In Table 6, we further test Svanberg problems (in different dimensions) for some infeasible initial points, and the stopping criterion threshold is  $\epsilon = 10^{-6}$  and  $\varphi_k = 0$ . The results show that our algorithm ALG 2.1 is always successful for all cases, and the iteration points can enter into the feasible set so faster. In view of NIO, NII and CPU, it follows that our algorithm is effective.

Table 5. *Numerical results for Experiment 2-I*

Prob	$n/m$	Initial point	Code	NI0	NII	NF0	NF	FV	CPU
Svanberg-10	10/30	$(0, 0, \dots, 0)^T$	ALG 2.1	0	16	17	1140	15.731517	0.34
			SNQP	0	28	28	1753	15.731533	—
			ALGO	0	15	21	1050	15.731517	—
			FSLE	0	36	227	258	15.731517	—
Svanberg-30	30/90	$(0, 0, \dots, 0)^T$	ALG 2.1	0	25	26	5490	49.142526	1.77
			SNQP	0	27	27	4975	49.142545	—
			ALGO	0	26	38	5670	49.142526	—
			FSLE	0	101	777	864	49.142526	—
Svanberg-50	50/150	$(0, 0, \dots, 0)^T$	ALG 2.1	0	33	34	11550	82.581912	5.91
			SNQP	0	37	37	11762	82.581928	—
			ALGO	0	35	51	12750	82.581912	—
			FSLE	0	108	881	968	82.581912	—
Svanberg-80	80/240	$(0, 0, \dots, 0)^T$	ALG 2.1	0	42	43	24720	132.749819	15.38
			SNQP	0	47	47	24100	132.749830	—
			ALGO	0	47	68	27360	132.749819	—
			FSLE	0	190	1666	1835	132.749819	—
Svanberg-100	100/300	$(0, 0, \dots, 0)^T$	ALG 2.1	0	46	91	53700	166.197172	26.38
			SNQP	0	46	46	27880	166.197199	—
			ALGO	0	53	66	35400	166.197171	—
			FSLE	0	178	1628	1782	166.197171	—

Table 6. *Numerical results for Experiment 2-II*

Prob	$n/m$	Initial point	NIO	NII	NF0	NF	FV	CPU
Svanberg-10	10/30	$(10, 10, \dots, 10)^T$	3	15	19	1320	15.731517	0.28
		$(-10, -10, \dots, -10)^T$	2	16	19	1440	15.731517	0.28
Svanberg-20	20/60	$(10, 10, \dots, 10)^T$	4	22	27	4380	32.427932	1.11
		$(-10, -10, \dots, -10)^T$	3	24	28	4560	32.427932	1.23
Svanberg-30	30/90	$(10, 10, \dots, 10)^T$	3	25	29	6480	49.142526	2.33
		$(-10, -10, \dots, -10)^T$	3	24	28	6030	49.142526	2.50
Svanberg-40	40/120	$(10, 10, \dots, 10)^T$	3	28	32	9480	65.861140	3.67
		$(-10, -10, \dots, -10)^T$	3	28	32	9480	65.861140	3.58
Svanberg-50	50/150	$(10, 10, \dots, 10)^T$	14	26	41	24150	82.581915	7.84
		$(-10, -10, \dots, -10)^T$	1	34	36	12900	82.581912	5.75
Svanberg-80	80/240	$(10, 10, \dots, 10)^T$	2	43	46	42720	132.749820	17.09
		$(5, 5, \dots, 5)^T$	2	47	50	63840	132.749824	19.53
Svanberg-100	100/300	$(10, 10, \dots, 10)^T$	3	43	47	57300	166.197173	26.81
		$(5, 5, \dots, 5)^T$	2	62	65	112500	166.197178	40.55
Svanberg-150	150/450	$(10, 10, \dots, 10)^T$	40	44	85	227700	249.818369	130.41
		$(5, 5, \dots, 5)^T$	3	62	66	123750	249.818369	96.16
Svanberg-200	200/600	$(10, 10, \dots, 10)^T$	4	78	83	219600	333.441310	279.95
		$(5, 5, \dots, 5)^T$	2	84	87	236400	333.441310	287.45
Svanberg-250	250/750	$(2, 2, \dots, 2)^T$	1	85	87	275250	417.064989	602.78
		$(3, 3, \dots, 3)^T$	1	90	92	281250	417.064989	593.48

**Experiment 3.** To show that ALG 2.1 performs cycle I for most of the iterations, we give Table 7 below relative to Tables 1 and 2 for test problems. The row labeled  $\sharp$  lists the problem number as given in Tables 1 and 2, and the column labeled  $\sharp$  lists the number of cycle I and cycle II performed by ALG 2.1, , i.e., N-cycle I and N-cycle II, respectively. The results reported in Table 7 are encouraging. Obviously, N-cycle I is much more than N-cycle II. Especially, for problem 12, ALG 2.1 always performs cycle I and does not perform cycle II. This also illustrate Remark 2.6 from the viewpoint of numerical results. Thus, the cost of computation for ALG 2.1 is relatively small.

Table 7. *Numerical results for performing Cycle I and Cycle II*

#	12	29	31	33	34	35	43	44	66	76	100	113	264
N-cycle I	20	8	15	9	9	6	12	7	52	17	49	11	22
N-cycle II	0	4	2	1	6	1	2	7	12	4	8	5	1

## 6. Concluding remarks

In this paper, we propose a new algorithm of combining (QP) subproblem with SLE for solving nonlinear inequality constrained optimization problems. The new algorithm starts from an arbitrarily initial iteration point. In order to ensure the global convergence of new algorithm, the search direction is obtained by a convex combination of the master direction and an auxiliary direction, which are solved by subproblem (QPs) and SLE (6), respectively. For overcoming the Maratos effect [5], a higher-order direction is obtained by solving another SLE (11). Moreover, the iteration points can always enter into the feasible set  $X$  and only one SLE need to be solved after a finite number of iterations. Using line search instead of arc search, our new algorithm possesses global and superlinear convergence under some mild assumptions without strict complementarity. Finally, some numerical results show that new algorithm is promising.

As a further work of this paper, the techniques introduced in this paper can be extended to solve general constrained optimization problems and minimax problems.

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